

Application of Explicit Symplectic Algorithms to Integration of Damping Oscillators

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Abstract

In this paper an approach is outlined. With this approach some explicit algorithms can be applied to solve the initial value problem of n -dimensional damped oscillators. This approach is based upon following structure: for any non-conservative classical mechanical system and arbitrary initial conditions, there exists a conservative system; both systems share one and only one common phase curve; and, the value of the Hamiltonian of the conservative system is, up to an additive constant, equal to the total energy of the non-conservative system on the aforementioned phase curve, the constant depending on the initial conditions. A key way applying explicit symplectic algorithms to damping oscillators is that by the Newton-Laplace principle the nonconservative force can be reasonably assumed to be equal to a function of a component of generalized coordinates q_i along a phase curve, such that the damping force can be represented as a function analogous to an elastic restoring force numerically in advance. Two numerical examples are given to demonstrate the good characteristics of the algorithms.

Keywords: Hamiltonian, dissipation, non-conservative system, damping, explicit symplectic algorithm

1. Introduction

Feng[1, 2, 3, 4], Marsden[5], Neri[6] and Yoshida[7] had developed a series of symplectic algorithms for Hamiltonian systems. These algorithms possess some

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advantages. But it is difficult to apply these algorithms to damping dynamical systems, because it has been stated in most classical textbooks that the Hamiltonian formalism focuses on solving conservative problems. Damping phenomena is very important in the modeling of dynamical systems, and can not be avoided. Our aim is to apply some explicit canonical algorithms to nonlinear damping dynamical systems, which is generated generally by FE-method. These canonical algorithms reported in this paper can be readily utilized for computing large-scale nonlinear damped dynamical systems.

Betch[8][9][10] attempted to apply directly some implicit algorithms to damping systems. The implicit symplectic algorithms utilized by Betch[8] possess a few good characteristic, e.g. energy-conservation, momentum-consistence, etc... In terms of energy-conservation, implicit symplectic algorithms might be better than explicit symplectic ones. But explicit symplectic schemes might be more suitable for nonlinear problems.

If one needs to apply symplectic algorithms to a dissipative system, one must convert the dissipative system into a Hamiltonian system or find some relationship between the dissipative system and a conservative one.

In the literature[11], we have stated a proposition describing a relation among a damping dynamical system and conservative ones:

Proposition 1.1. *For any non-conservative classical mechanical system and arbitrary initial condition, there exists a conservative system; both systems sharing one and only one common phase curve; and the value of the Hamiltonian of the conservative system is equal to the sum of the total energy of the non-conservative system on the aforementioned phase curve and a constant depending on the initial condition.*

In other words, a dissipative ordinary equation and a conservative equation may possess a common particular solution. In the next section, an analytical examples are given to explain this proposition. Readers can find the detailed proof of Proposition 1.1 in the reference[11]

In the Literature [12] a basic explicit canonical integrator is proposed. Based on this basic scheme, Neri[6] constructed 4-order explicit canonical integrator, and then Yoshida [7] proposed a general method to construct higher order explicit symplectic integrator. Utilizing the Proposition 1.1, we apply this class of explicit canonical integrators to damping dynamical systems. This point will be in detail stated in sec. 3.

2. One-dimensional Analytical Example

Consider a special one-dimensional simple mechanical system:

$$\ddot{x} + c\dot{x} = 0, \quad (1)$$

where c is a constant. The exact solution of the equation above is

$$x = A_1 + A_2 e^{-ct}, \quad (2)$$

where A_1, A_2 are constants. Differentiation gives the velocity:

$$\dot{x} = -cA_2 e^{-ct}. \quad (3)$$

From the initial condition x_0, \dot{x}_0 , we find $A_1 = x_0 + \dot{x}_0/c, A_2 = -\dot{x}_0/c$. Inverting Eq. (2) yields

$$t = -\frac{1}{c} \ln \frac{x - A_1}{A_2} \quad (4)$$

and by substituting into Eq. (3), such we have

$$\dot{x} = -c(x - A_1) \quad (5)$$

The dissipative force F in the dissipative system (1) is

$$F = c\dot{x}. \quad (6)$$

Substituting Eq. (5) into Eq. (6), the conservative force \mathcal{F} is expressed as

$$\mathcal{F} = -c^2(x - A_1); \quad (7)$$

Clearly, the conservative force \mathcal{F} depends on the initial condition of the dissipative system (1), in other words, an initial condition determines a conservative force. Consequently, a new conservative system yields

$$\ddot{x} + \mathcal{F} = 0 \rightarrow \ddot{x} - c^2(x - A_1) = 0. \quad (8)$$

The stiffness coefficient in this equation must be negative. One can readily verify that the particular solution (2) of the dissipative system can satisfy the conservative one (8). This point agrees with Proposition (1.1).

The potential of the conservative system (8) is

$$V = \int_0^x [-c^2(x - A_1)] dx = -\frac{c^2}{2}x^2 + c^2A_1x$$

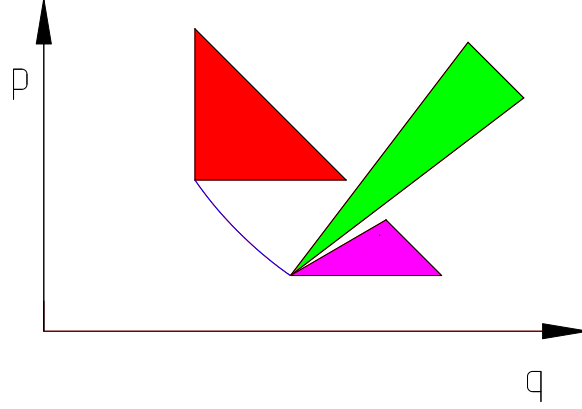


Figure 1: Relationship between nonconservative system (1) and conservative one (8)

Therefore the Hamiltonian is

$$\hat{H} = T + V = \frac{1}{2}p^2 - \frac{c^2}{2}x^2 + c^2A_1x,$$

where $p = \dot{x}$.

Furthermore, Proposition (1.1) can be depicted by Fig. 2. The phase flow of conservative system (2) transforms the red area in phase space to the purple area; the phase flow of conservative system (8) transforms the red area to the green area. The blue curve in Fig. 2 illustrates the common phase curve. If one draws more common phase curves and phase flows, the picture will like a flower, the phase flow of the nonconservative system likes a pistil and phase flows conservative systems like petals.

3. Modification Symplectic Numerical Schemes

3.1. Basic Explicit Symplectic Numerical Schemes

In the paper[12][6][7] a symplectic algorithm based second kind generation function was stated

$$\begin{aligned} \mathbf{p}^{i+1} &= \mathbf{p}^i - \tau H_q(\mathbf{p}^{i+1}, \mathbf{q}^i) \\ \mathbf{q}^{i+1} &= \mathbf{q}^i + \tau H_p(\mathbf{p}^{i+1}, \mathbf{q}^i), \end{aligned} \tag{9}$$

where the superscript i denotes the i -th time node, \mathbf{q} denotes coordinates and \mathbf{p} denotes canonical momenta, and H denotes Hamiltonian quantity, $H_q = \partial H / \partial \mathbf{q}$, $H_p =$

$\partial H / \partial \mathbf{p}$. If the Hamiltonian is separable, i.e. $H = U(\mathbf{p}) + V(\mathbf{q})$, $V_q = H_q$, $U_p = H_p$, then the symplectic scheme(9) above becomes an explicit symplectic scheme:

$$\begin{aligned} \mathbf{p}^{i+1} &= \mathbf{p}^i - \tau V_q(\mathbf{q}^i) \\ \mathbf{q}^{i+1} &= \mathbf{q}^i + \tau U_p(\mathbf{p}^{i+1}). \end{aligned} \quad (10)$$

For some nonlinear vibration mechanical system, $V_q = K(\mathbf{q})\mathbf{q}$.

Let us consider an n -dimensional nonlinear oscillator:

$$\ddot{\mathbf{q}} + C\dot{\mathbf{q}} + K\mathbf{q} = 0, \quad (11)$$

where C denotes a non-linear damping coefficient matrix which depends on \mathbf{q} , and K denotes a non-linear stiffness matrix which depends on \mathbf{q} and consists of two parts $K = \check{K} + \hat{K}$ (\check{K} is a diagonal matrix).

In accordance with Proposition 1.1, a conservative mechanical system was found associated with the dissipative system (11) in addition to its initial conditions. Subject to these initial conditions, the dissipative system (11) possesses a common phase curve γ with the conservative system. As in Eq. (7), we can consider that the components of the damping force $C\dot{\mathbf{q}}$ determine the components of a conservative force on the phase curve γ

$$\begin{aligned} c_{11}\dot{q}_1 &= \rho_{11}(q_1) & \dots & & c_{1n}\dot{q}_n &= \rho_{1n}(q_1) \\ \vdots & & \ddots & & \vdots & \\ c_{n1}\dot{q}_1 &= \rho_{21}(q_n) & \dots & & c_{nn}\dot{q}_n &= \rho_{nn}(q_n). \end{aligned} \quad (12)$$

For convenience, this conservative force is assumed to be an elastic restoring force:

$$\begin{aligned} \rho_{11}(q_1) &= \kappa_{11}(q_1)q_1 & \dots & & \rho_{1n}(q_1) &= \kappa_{1n}(q_1)q_1 \\ \vdots & & \ddots & & \vdots & \\ \rho_{n1}(q_1) &= \kappa_{n1}(q_n)q_n & \dots & & \rho_{nn}(q_n) &= \kappa_{nn}(q_n)q_n. \end{aligned} \quad (13)$$

In a similar manner, the components of the non-conservative force $\hat{K}\mathbf{q}$ are equal to the components of a conservative force on the phase curve γ

$$\begin{aligned} \hat{K}_{11}q_1 &= \chi_{11}(q_1) & \dots & & \hat{K}_{1n}q_n &= \chi_{1n}(q_1) \\ \vdots & & \ddots & & \vdots & \\ \hat{K}_{n1}q_1 &= \chi_{21}(q_n) & \dots & & \hat{K}_{nn}q_n &= \chi_{nn}(q_n). \end{aligned} \quad (14)$$

The conservative force can likewise be assumed to an elastic restoring force:

$$\begin{aligned} \chi_{11}(q_1) &= \lambda_{11}(q_1)q_1 & \dots & & \chi_{1n}(q_1) &= \lambda_{1n}(q_1)q_1 \\ \vdots & & \ddots & & \vdots & \\ \chi_{n1}(q_1) &= \lambda_{n1}(q_n)q_n & \dots & & \chi_{nn}(q_n) &= \lambda_{nn}(q_n)q_n. \end{aligned} \quad (15)$$

By an appropriate transformation, an equivalent stiffness matrix \tilde{K} that is diagonal in form can be obtained

$$\tilde{K}_{ii} = \sum_{l=1}^n \kappa_{il}(q_l) + \lambda_{il}(q_l). \quad (16)$$

Consequently, an n -dimensional conservative system is obtained

$$\ddot{\mathbf{q}} + (\check{K} + \tilde{K})\mathbf{q} = 0 \quad (17)$$

which shares the common phase curve γ with the n -dimensional damping system described by (11). In this paper, the conservative system is called the 'substitute' conservative system. The Lagrangian of Eqs.(17) is

$$\hat{L} = \frac{1}{2} \dot{\mathbf{q}}^T \dot{\mathbf{q}} - \int_0^{\mathbf{q}} (\check{K}\mathbf{q})^T d\mathbf{q} - \int_0^{\mathbf{q}} (\tilde{K}\mathbf{q})^T d\mathbf{q}, \quad (18)$$

with the Hamiltonian

$$\hat{H} = \frac{1}{2} \mathbf{p}^T \mathbf{p} + \int_0^{\mathbf{q}} (\check{K}\mathbf{q})^T d\mathbf{q} + \int_0^{\mathbf{q}} (\tilde{K}\mathbf{q})^T d\mathbf{q}, \quad (19)$$

where $\mathbf{0}$ is the zero vector, and $\mathbf{p} = \dot{\mathbf{q}}$. Here \hat{H} in Eq. (19) is the mechanical energy of the conservative system (17), because $\int_0^{\mathbf{q}} (\tilde{K}\mathbf{q})^T d\mathbf{q}$ is a potential function such that \hat{H} is independent of the path taken in phase space.

Subject to a certain initial condition, one need merely to solve the conservative system(17). But one must in advance obtain the numerical approximation of the matrix \tilde{K} for a time step, such that one can utilize the algorithm (10) to integrate the conservative system (17) for a time step. One can repeat this process above up to the end. In this way one obtains the numerical particular solution of the conservative system (17), which is exactly the numerical particular solution of the damping one. The numerical approximation of the matrix \tilde{K} can be assumed as:

$$\tilde{K} = \begin{bmatrix} \tilde{K}_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \tilde{K}_{nn} \end{bmatrix} \quad (20)$$

$$\tilde{K}_j(q_j^i) = c_{jl}\dot{q}_l^i/q_j^i + \hat{K}_{jl}q_l^i/q_j^i$$

Hence the explicit canonical scheme (10) can be modified into

$$\begin{aligned} \tilde{K}_j^i(q_j^i) &= c_{jl}\dot{q}_l^i/q_j^i + \hat{K}_{jl}q_l^i/q_j^i \\ p_j^{i+1} &= p_j^i - \tau[K_j + \tilde{K}_j^i(q_j^i)]q_j^i \\ q_j^{i+1} &= q_j^i + \tau p_j^{i+1} \end{aligned} \quad (21)$$

The scheme above is a one order scheme. Furthermore one can construct higher order explicit canonical schemes utilizing the method reported in the literatures[6][7]. Now consider a map from $\mathbf{z} = \mathbf{z}(0)$ to $\mathbf{z}' = \mathbf{z}(\tau)$:

$$\mathbf{z}' \approx \left(\prod_{i=1}^h e^{r_i \tau E} e^{s_i \tau F} + O(\tau^{n+1}) \right) \mathbf{z}, \quad (22)$$

where

$$\mathbf{z} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}, \mathbf{z}' = \begin{bmatrix} \mathbf{p}' \\ \mathbf{q}' \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} F = \begin{bmatrix} 0 & -(K + \tilde{K}) \\ 0 & 0 \end{bmatrix}.$$

In fact Eq.(22) is the succession of the following mappings,

$$\begin{aligned} \mathbf{p}^{j+1} &= \mathbf{p}^j - s_i \tau V_q(\mathbf{q}^j) \\ \mathbf{q}^{j+1} &= \mathbf{q}^j + r_i \tau U_p(\mathbf{p}^{j+1}) \end{aligned} \quad (23)$$

In reality the difference between the equations above and Eq.(21) is that the coefficients s_i, r_i before the time step τ . In the literature [7] a generalized method to determine s_i, r_i were given. Therefore, the higher order explicit canonical scheme can be represented as:

$$\tilde{K}(\mathbf{q}^j) = \begin{bmatrix} \tilde{K}_1(\mathbf{q}_1^j) & & 0 \\ & \ddots & \\ 0 & & \tilde{K}_n(\mathbf{q}_n^j) \end{bmatrix} \tilde{K}_\alpha(\mathbf{q}_\alpha^j) = \sum_{l=1}^n c_{\alpha l} \dot{q}_l^j / q_\alpha^j + \hat{K}_{\alpha l} q_l^j / q_\alpha^j$$

$$E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} F = \begin{bmatrix} 0 & -(K + \tilde{K}) \\ 0 & 0 \end{bmatrix} \quad (24)$$

$$\mathbf{z}^{j+1} = \left(\prod_{i=1}^h e^{s_i \tau F} e^{r_i \tau E} \right) \mathbf{z}^j$$

4. Numerical Examples

Two examples will be given to shown this numerical method24.

4.1. The First Example

To begin, we consider a Van Der Pol's oscillator

$$\ddot{x} + \mu \dot{x}(x^2 - 1) + x = 0, \quad (25)$$

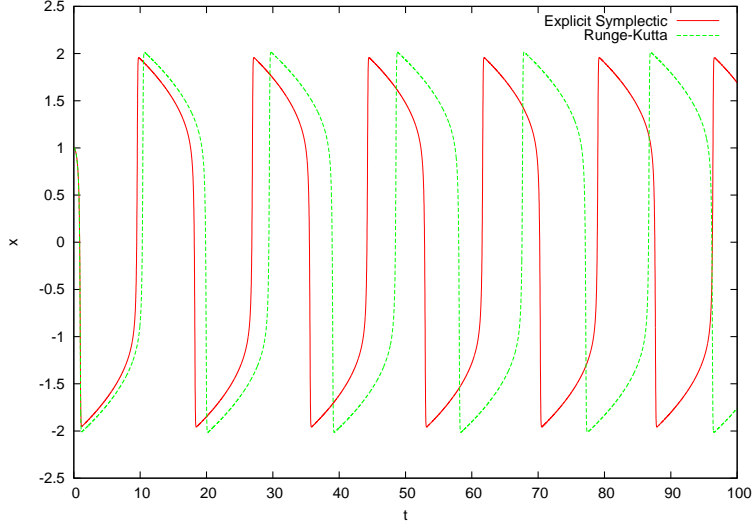


Figure 2: The resonance of the Van Der Pol's oscillator

where $\mu = 10$. The initial conditions are given by $x_0 = 1$, $\dot{x} = 0$. We employ the 4-order explicit symplectic method (24) with coefficients

$$s_1 = s_4 = [2 + (\sqrt[3]{2} + 1/\sqrt[3]{2})]/6, \quad s_2 = s_3 = [1 - (\sqrt[3]{2} + 1/\sqrt[3]{2})]/6,$$

$$r_1 = r_3 = [2 + (\sqrt[3]{2} + 1/\sqrt[3]{2})]/3, \quad r_2 = -[2 + (\sqrt[3]{2} + 1/\sqrt[3]{2})]/3, \quad r_4 = 0,$$

and classical explicit 4-order Runge-Kutta method to compute the resonance of the Van Der Pol's oscillator (26) respectively, then employ a same method to integrate the results to the total energy, which is the sum of the mechanical energy and the work done by damping forces in the system (25). The both methods are run with the same step size $\tau = 0.01$. The resonance is shown in Fig. 2, and the total energy is shown in Fig. 3.

It is apparent from Fig. 3 that the explicit symplectic method (24) has qualitatively different behavior to the Runge-Kutta method. The energy divergence between the explicit symplectic method and the exact solution is smaller than that between Runge-Kutta method and the exact solution. The energy divergence between the explicit symplectic method and Runge-Kutta method increases with the time evolution. Due to the increase of the energy, the phase difference between both the results in Fig. 2 increases also with the time evolution.

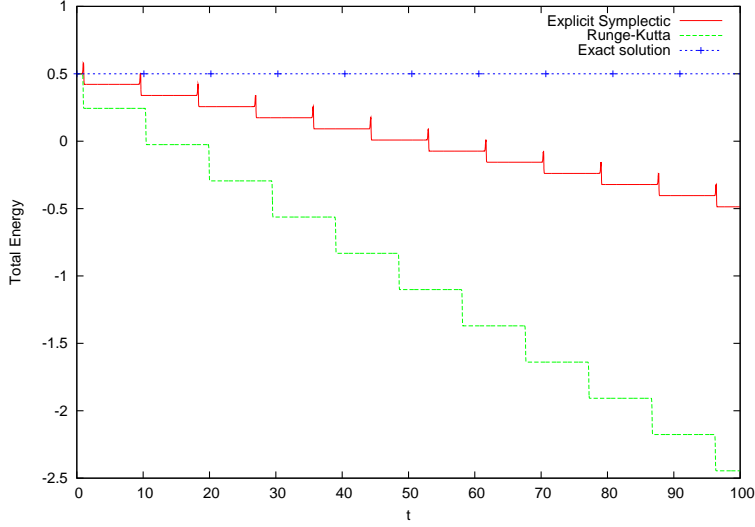


Figure 3: The total energy of the Van Der Pol's oscillator

4.2. The Second Example

In the second example, we consider a 2-dimensional damped nonlinear Duffing oscillator

$$\begin{aligned} 2\ddot{q}_1 + 0.1\dot{q}_1 + (2 + 0.1q_1^2)q_1 + q_2 &= 0 \\ 3\ddot{q}_2 + 0.2\dot{q}_2 + q_1 + (2 + 0.2q_2^2)q_2 &= 0, \end{aligned} \quad (26)$$

with the initial conditions $q_1 = 0$, $q_2 = 0$, $\dot{q}_1 = 0$, $\dot{q}_2 = 1$. The program of the both methods with the step size $\tau = 0.01$ are carried out to simulate Eq. (26). The resonance is shown in Fig. 2, the numerical solution of the total energy is shown in Fig.5.

There is only tiny difference between resonance results of the two methods, correspondingly, the difference among the total energy obtained by the numerical methods and analytical methods is very tiny. As numerical examples in the other literatures[13], that explicit Runge-Kutta method must cause numerical pseudo dissipation which might be positive or negative. The difference between our numerical examples and the examples in the literature[13] is the total energy in our examples and the mechanical energy in their examples¹.

¹Fig.6.1 in the literature[13]

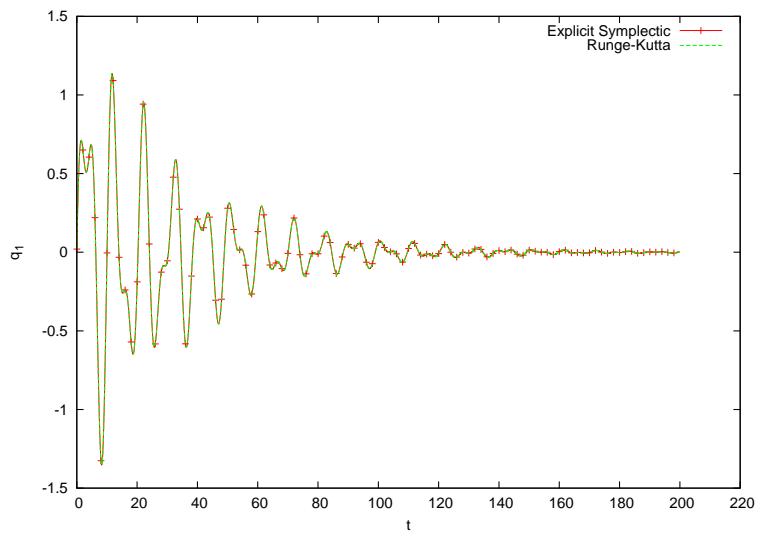


Figure 4: The 1-th displacement of the damped Duffing oscillator

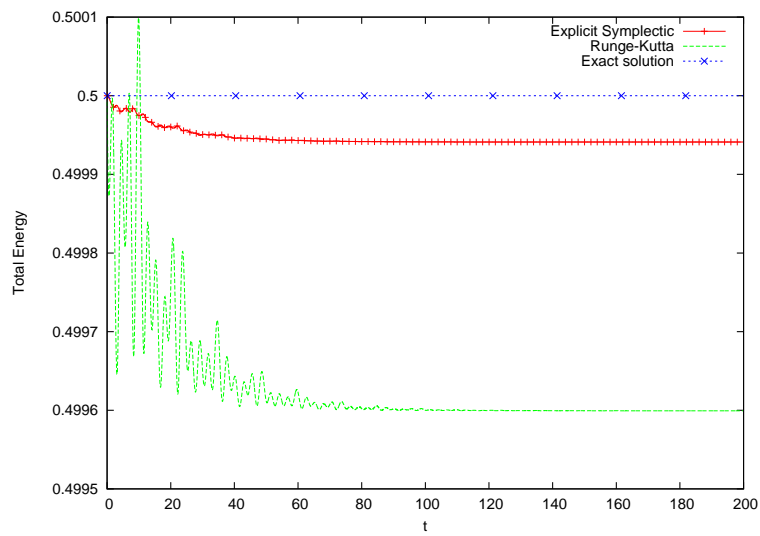


Figure 5: Total energy of the damped dissipative oscillator

5. Conclusions

We have introduced a class of explicit symplectic algorithms to dissipative mechanical systems successfully, by changing these algorithms into the scheme.(24). Because the algorithms (24) are explicit and possess good energy preserving characteristics, the explicit symplectic algorithms (24) is quite suitable for long term integration of arbitrary dimensional nonlinear dissipative mechanical systems.

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